J. Fluid Mech. (1983), vol. 129, pp. 337–345
Printed in Great Britain

Self-consistent calculation of the α-effect and turbulent magnetic diffusion

By I. T. DRUMMOND

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW

(Received 2 August 1982)

We use the method of Phythian & Curtis (1978) to obtain a self-consistent calculation, in lowest order of perturbation theory, for the α-coefficient and effective diffusivity of a magnetic field in a plasma with Gaussian turbulence.

1. Introduction

In this paper we extend to the case of a magnetic field in a turbulent plasma, the ideas of Phythian & Curtis (1978) for calculating the effective diffusivity of a scalar field in Gaussian turbulence. Because of the vector character of the magnetic field, this problem is inherently more complicated than the scalar field case. There are in fact two parameters to be calculated, the effective diffusivity and the α-parameter (Steenbeck, Krause & Rädler 1966; Roberts & Stix 1971; Moffatt 1979). Nevertheless the calculation can be given a formulation similar to the scheme of Phythian & Curtis (1978).

However, although the formal structure of the calculation is the same as for the scalar field case, the physical significance of the self-consistent parameters must be established with some care.

2. Perturbation series for the effective Green function

The equation governing the evolution of a magnetic field in a plasma with velocity distribution $\boldsymbol{u}(\boldsymbol{x}, t)$ is

$$\frac{\partial}{\partial t}\boldsymbol{B} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) - \eta_0 \nabla \wedge (\nabla \wedge \boldsymbol{B}), \tag{2.1}$$

where η_0 is the molecular diffusivity of the plasma. We shall assume that the plasma is incompressible and that the turbulence is homogeneous, isotropic and subject to Gaussian statistics.

Following Phythian & Curtis (1978), we anticipate the existence of the quantities we wish to calculate, namely an effective diffusivity η and an α-parameter, and rewrite (2.1) in the form

$$\frac{\partial}{\partial t}\boldsymbol{B} = \alpha \nabla \wedge \boldsymbol{B} - \eta \nabla \wedge (\nabla \wedge \boldsymbol{B}) + \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) - (\mu_2 + \mu_4 + \dots)\nabla \wedge (\nabla \wedge \boldsymbol{B}) + (\gamma_2 + \gamma_4 + \dots)\nabla \wedge \boldsymbol{B}, \tag{2.2}$$

where

$$\eta - \eta_0 = -(\mu_2 + \mu_4 + \dots), \tag{2.3}$$

$$\alpha = -(\gamma_2 + \gamma_4 + \dots). \tag{2.4}$$

Here μ_{2n} and γ_{2n} are quantities $O((\boldsymbol{u})^{2n})$. Finally we use the fact that $\nabla \cdot \boldsymbol{B} = 0$ to obtain from (2.2)

$$\frac{\partial}{\partial t}\boldsymbol{B} - \alpha \nabla \wedge \boldsymbol{B} - \eta \nabla^2 \boldsymbol{B} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) - (\mu_2 + \ldots) \nabla \wedge (\nabla \wedge \boldsymbol{B}) + (\gamma_2 + \ldots) \nabla \wedge \boldsymbol{B}. \tag{2.5}$$

Let the Green function (or the magnetic propagator) for the above equation be $G_{ij}(\boldsymbol{x}, t \,|\, \boldsymbol{x}', t')$. It satisfies

$$G_{ij}(\boldsymbol{x}, t \,|\, \boldsymbol{x}', t') = \begin{cases} 0 & (t < t'), \\ \delta_{ij}\, \delta(\boldsymbol{x} - \boldsymbol{x}') & (t = t'), \end{cases} \tag{2.6}$$

and for $t > t'$ obeys

$$\left[\left(\frac{\partial}{\partial t} - \eta \nabla^2\right) \delta_{ij} - \alpha \epsilon_{iml}\, \partial_m\right] G_{lj}(\boldsymbol{x}, t \,|\, \boldsymbol{x}', t')$$
$$= [K_{imrl}(\partial_m u_r - (\mu_2 + \ldots)\, \partial_m \partial_r) + (\gamma_2 + \ldots)\, \epsilon_{iml}]\, G_{lj}(\boldsymbol{x}, t \,|\, \boldsymbol{x}', t'), \tag{2.7}$$

where

$$K_{imrl} = \epsilon_{imn}\, \epsilon_{nrl}. \tag{2.8}$$

To solve (2.7) we introduce $G^{(0)}_{ij}$, which coincides with G_{ij} for $t \leqslant t'$, and, for $t > t'$, obeys

$$\left[\left(\frac{\partial}{\partial t} - \eta \nabla^2\right) \delta_{il} - \alpha \epsilon_{iml}\, \partial_m\right] G^{(0)}_{lj}\,(\boldsymbol{x} - \boldsymbol{x}', t - t') = 0. \tag{2.9}$$

We then find

$$G_{ij}(\boldsymbol{x}, t \,|\, \boldsymbol{x}', t') = G^{(0)}_{ij}(\boldsymbol{x} - \boldsymbol{x}', t - t') + \int \mathrm{d}^3 \boldsymbol{x}''\, \mathrm{d}t''\, G^{(0)}_{il}(\boldsymbol{x} - \boldsymbol{x}'', t - t'')$$
$$\times [K_{lmrl'}(\partial''_m u_r(\boldsymbol{x}'', t'') - (\mu_2 + \ldots)\, \partial''_m \partial''_r$$
$$+ (\gamma_2 + \ldots)\, \epsilon_{lml'}\, \partial''_m]\, G_{l'j}(\boldsymbol{x}'', t'' \,|\, \boldsymbol{x}', t'). \tag{2.10}$$

From this equation we can develop a perturbation series for G_{ij} in powers of the velocity field. On averaging the series over the ensemble of velocity fields we obtain a corresponding series for the effective magnetic propagator (EMP) $\mathscr{G}_{ij}$, defined by

$$\mathscr{G}_{ij}(\boldsymbol{x} - \boldsymbol{x}', t - t') = \langle G_{ij}(\boldsymbol{x}, t \,|\, \boldsymbol{x}', t') \rangle. \tag{2.11}$$

To $O((\boldsymbol{u})^2)$ this series is

$$\mathscr{G}_{ij}(\boldsymbol{x} - \boldsymbol{x}', t - t')$$
$$= G^{(0)}_{ij}(\boldsymbol{x} - \boldsymbol{x}', t - t')$$
$$+ \int \mathrm{d}^3 \boldsymbol{x}''\, \mathrm{d}t''\, G^{(0)}_{il}(\boldsymbol{x} - \boldsymbol{x}'', t - t'')\, [-\mu_2 K_{lmrl'}\, \partial''_m \partial''_r + \gamma_2\, \epsilon_{lml'}\, \partial''_m]\, G^{(0)}_{l'j}(\boldsymbol{x}'' - \boldsymbol{x}', t'' - t')$$
$$+ \int \mathrm{d}^3 \boldsymbol{x}''\, \mathrm{d}t''\, \mathrm{d}^3 \boldsymbol{x}'''\, \mathrm{d}t'''\, K_{lmrp}\, K_{p'm'r'l'}\; G^{(0)}_{il}(\boldsymbol{x} - \boldsymbol{x}'', t - t'')$$
$$\times\, \partial''_m \partial'''_{m'} \langle u_r(\boldsymbol{x}'', t'')\, u_{r'}(\boldsymbol{x}''', t''') \rangle\, G^{(0)}_{pp'}(\boldsymbol{x}'' - \boldsymbol{x}''', t'' - t''')\, G^{(0)}_{l'j}\,(\boldsymbol{x}''' - \boldsymbol{x}', t''' - t'). \tag{2.12}$$

It is possible to associate these and the higher-order terms in the expansion with certain diagrams. However, it is easier to explain the significance of these diagrams in transform space. We discuss this in §3.

3. Transform space and perturbation-theory rules

The homogeneity in space makes it natural to take a Fourier transform in the relative spatial variable, while the fact that certain quantities blow up exponentially in the relative time variable makes it natural to use the Laplace transform in this case.

We define

$$\bar{G}_{ij}^{(0)}(\boldsymbol{k},s) = \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \int_{t'}^{\infty} \mathrm{d}t\, \mathrm{e}^{-s(t-t')}\, G_{ij}^{(0)}(\boldsymbol{x}-\boldsymbol{x}', t-t'). \tag{3.1}$$

Equation (2.9) together with the boundary condition on $G_{ij}^{(0)}$ implies that

$$[(s+\eta k^2)\,\delta_{il} - \mathrm{i}\alpha\epsilon_{i\mu l}\,\kappa_{\mathrm{m}}]\,\bar{G}_{lj}^{(0)}(\boldsymbol{k},s) = \delta_{ij}. \tag{3.2}$$

That is

$$\bar{G}_{ij}^{(0)}(\boldsymbol{k},s) = \left[\delta_{ij} - \frac{k_i k_j}{k^2}\right] A(k^2,s) - \mathrm{i}\epsilon_{imj}\, k_m\, B(k^2,s) + \frac{k_i k_j}{k^2}\frac{1}{s+\eta k^2}, \tag{3.3}$$

where

$$A(k^2,s) = \frac{s+\eta k^2}{(s+\eta k^2)^2 - \alpha^2 k^2}, \tag{3.4}$$

$$B(k^2,s) = \frac{\alpha}{(s+\eta k^2)^2 - \alpha^2 k^2}. \tag{3.5}$$

It turns out that when $G_{ij}^{(0)}(\boldsymbol{k},x)$ is used in constructing perturbation-theory terms the last term in (3.3) yields a vanishing contribution. When we modify $G_{ij}^{(0)}$ by omitting this term is satisfies the useful condition,

$$k_i\, \bar{G}_{ij}^{(0)}(\boldsymbol{k},s) = k_j\, \bar{G}_{ij}^{(0)}(\boldsymbol{k},s) = 0. \tag{3.6}$$

The transform of the velocity correlation function is

$$F_{ij}(\boldsymbol{k},s) = \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \int_{t'}^{\infty} \mathrm{d}t\, \mathrm{e}^{-s(t-t')} \langle u_i(\boldsymbol{x},t)\, u_j(\boldsymbol{x}',t')\rangle. \tag{3.7}$$

The incompressibility of the flow implies that

$$F_{ij}(\boldsymbol{k},s) = (k^2\delta_{ij} - k_i k_j)\, \Phi(k^2,s) + \mathrm{i}\epsilon_{ilj}\, k_l\, \Psi(k^2,s). \tag{3.8}$$

We have

$$v^2 = 2\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \int \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, \Phi(k^2,s)\, k^2, \tag{3.9}$$

$$h = 2\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \int \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, \Psi(k^2,s)\, k^2, \tag{3.10}$$

where v is the r.m.s. velocity and h is the mean helicity density. Here the s-integration runs in an imaginary direction along a contour lying to the right of the singularities of Φ or Ψ in the s-plane.

The rules for writing down perturbation-theory contributions to $\bar{\mathscr{G}}_{ij}(\boldsymbol{k},s)$, the transform of the EMP, are similar in spirit to those explained by Phythian & Curtis (1978) for the scalar case. Each contribution is associated with a graph. The lowest-order graphs are illustrated in figure 1. Fourth-order graphs are shown in figure 2. The rules are as follows.

Each line in a graph has assigned to it a Fourier and a Laplace variable, $\boldsymbol{k}$ and

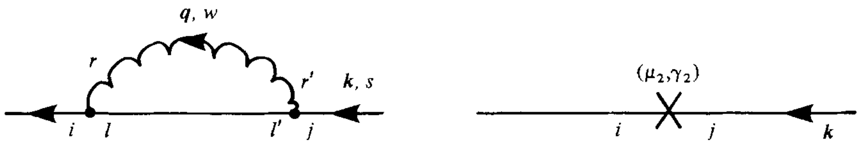

FIGURE 1. Graphs for lowest-order calculation.

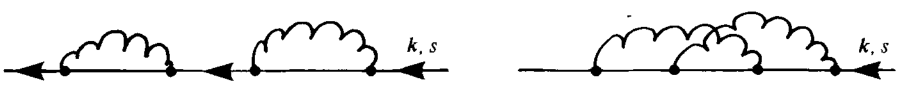

FIGURE 2. Examples of two-loop graphs.

s respectively. The sum of all the variables flowing into a vertex is zero. The lines and vertices of each graph are associated with factors in the integrand for the perturbation-theory term as shown in figure 3. Finally the independent (loop) variables $\boldsymbol{k}$, s are each integrated with a weight

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \int \frac{ds}{2\pi i}. \tag{3.11}$$

The s-integration contour runs in an imaginary direction along a path disposed appropriately relative to the singularities in the s-plane. We shall see this worked out in detail in the lowest-order calculation.

Again following Phythian & Curtis (1978), we organize the terms in the perturbation series in a standard way so that we can write

$$\overline{\mathscr{G}}_{ij}(\boldsymbol{k}, s) = \overline{G}^{(0)}_{ij}(\boldsymbol{k}, s) + \overline{G}^{(0)}_{il}(\boldsymbol{k}, s)\,\Sigma_{ll'}(\boldsymbol{k}, s)\,\overline{\mathscr{G}}_{l'j}(\boldsymbol{k}, s), \tag{3.12}$$

where $\Sigma(\boldsymbol{k}, s)$ is a sum over all irreducible bubbles (that is, those graphs that cannot be divided into two by cutting one line). To lowest order $\Sigma(\boldsymbol{k}, s)$ is obtained from the graphs in figure 1 after the external propagator factors $G^{(0)}(\boldsymbol{k}, s)$ have been removed.

We have then

$$\Sigma_{ij}(\boldsymbol{k}, s) = \Sigma^{(a)}_{ij}(\boldsymbol{k}, s) + \Sigma^{(b)}_{ij}(\boldsymbol{k}, s), \tag{3.13}$$

where

$$\Sigma^{(a)}_{ij}(\boldsymbol{k}, s) = i^2 \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \int \frac{dw}{2\pi i} F_{rr'}(\boldsymbol{q}, w)\,\overline{G}^{(0)}_{ll'}(\boldsymbol{k}-\boldsymbol{q}, s-w)$$

$$\times K_{imrl}\,K_{l'm'r'j}\,k_m\,(k-q)_{m'}, \tag{3.14}$$

$$\Sigma^{(b)}_{ij}(\boldsymbol{k}, s) = -\mu_2(k^2\delta_{ij} - k_i k_j) + i\gamma_2\,\epsilon_{imj}\,k_m. \tag{3.15}$$

We note that

$$k_i\,\Sigma^{(a)}_{ij}(\boldsymbol{k}, s) = 0; \tag{3.16}$$

therefore

$$\Sigma^{(a)}_{ij}(\boldsymbol{k}, s) = (k^2\delta_{ij} - k_i k_j)\,C(k^2, s) - i\epsilon_{imj}\,k_m\,D(k^2, s). \tag{3.17}$$

The expression for C and D may be obtained from (3.14). We are particularly interested in the limit $\boldsymbol{k} \to 0$. We find then (see (3.3) and (3.8))

$$C(0, s) = -\frac{2}{3} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \int \frac{dw}{2\pi i} q^2[\Phi(q^2, w)\,A(q^2, s-w) + \Psi(q^2, w)\,B(q^2, s-w)], \tag{3.18}$$

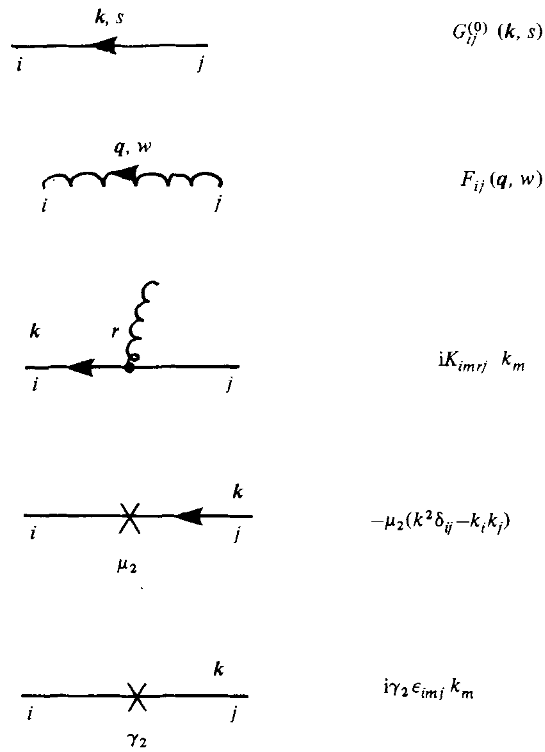

FIGURE 3. Rules for vertices and propagators.

$$D(0,s) = \frac{2}{3}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\int\frac{\mathrm{d}w}{2\pi\mathrm{i}}q^2[q^2\Phi(q^2,w)\,B(q^2,s-w)+\Psi(q^2,w)\,A(q^2,s-w)]. \quad (3.19)$$

We can now conveniently discuss the disposition of the w-contour. Assume for the sake of argument that the velocity correlation function dies away $\propto \mathrm{e}^{-\omega_0|t-t'|}$, then the rightmost singularity of $\Phi(q^2, w)$ and $\Psi(q^2, w)$ will lie at $w = -\omega_0$. From (3.4) and (3.5) it is clear that the rightmost singularity of $A(q^2, s-w)$ and $B(q^2, s-w)$ is to the left of $s-w = \frac{1}{2}\alpha^2$. It follows that for

$$\operatorname{Re} s > \frac{\alpha^2}{2\eta} - \omega_0, \quad (3.20)$$

the integrands in (3.18) and (3.19) have a strip of analyticity

$$-\omega_0 < \operatorname{Re} w < \operatorname{Re} s - \frac{\alpha^2}{2\eta}. \quad (3.21)$$

It is in this strip that we place the w-integration contour.

When formulating the self-consistent equations for α and η, it will be necessary to assume that $C(0,s)$ and $D(0,s)$ are analytic in a region including the origin. From (3.20) we see that this imposes a consistency requirement that

$$\frac{\alpha^2}{2\eta} < \omega_0. \quad (3.22)$$

4. Self-consistency equations

Equation (3.12) implies that

$$\bar{\mathscr{G}}_{ij}^{-1}(\boldsymbol{k},s) = \bar{G}_{ij}^{(0)-1}(\boldsymbol{k},s) - \Sigma_{ij}(\boldsymbol{k},s), \tag{4.1}$$

where the exponent refers to matrix inversion. That is, to $O((\boldsymbol{u})^2)$,

$$\bar{\mathscr{G}}_{ij}^{-1}(\boldsymbol{k},s) = (s+\eta k^2)\,\delta_{ij} - \mathrm{i}\alpha\epsilon_{imj}\,k_m - (k^2\delta_{ij}-k_i\,k_j)\,(C(k^2,s) - \mu_2) + \mathrm{i}\epsilon_{imj}\,k_m(D(k^2,s)-\gamma_2). \tag{4.2}$$

We see then that if we choose μ_2 and γ_2 so that

$$\mu_2 = C(0,0), \tag{4.3}$$

$$\gamma_2 = D(0,0), \tag{4.4}$$

then the terms $O(l)$ and $O(k^2)$ in $\bar{\mathscr{G}}_{ij}^{-1}(k,s)$ are correctly given for small s by the parameters α and η respectively. Provided we are interested in long time and distance scales these are also the parameters we expect to be physically significant.

To $O((\boldsymbol{u})^2)$ then, the self-consistent equations for α and η are

$$\eta-\eta_0 = \frac{2}{3}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\int\frac{\mathrm{d}w}{2\pi\mathrm{i}}\,q^2[\Phi(q^2,w)\,A(q^2,-w)+\Psi(q^2,w)\,B(q^2,-w)], \tag{4.5}$$

$$\alpha = -\frac{2}{3}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\int\frac{\mathrm{d}w}{2\pi\mathrm{i}}\,q^2,w)\,B(q^2,-w)+\Psi(q^2,w)\,A(q^2,-w)]. \tag{4.6}$$

It is implicit in the above argument that there are no singularities in the s-plane near $s=0$ to vitiate the assumption of smoothness in this neighbourhood. We therefore expect a constraint like (3.22) to hold for our solutions of (4.5) and (4.6).

In order to obtain tractable equations we shall again follow Phythian & Curtis and assume that the time dependence of the velocity correlation function is given entirely by a factor $\mathrm{e}^{-\omega_0|t-t'|}$. In that case Φ and Ψ have the forms

$$\Phi(q^2,w) = \frac{\phi(q^2)}{w+\omega_0}, \tag{4.7}$$

$$\Psi(q^2,w) = \frac{\psi(q^2)}{w+\omega_0}. \tag{4.8}$$

We will assume further that ϕ and ψ have their weight concentrated on a shell $q=k_0$ (Kraichnan 1970). That is

$$\phi(q^2) \sim \psi(q^2) \propto \delta(q-k_0). \tag{4.9}$$

The self-consistent equations become

$$\eta-\eta_0 = \frac{1}{3}\frac{v^2(\omega_0+\eta k_0^2)+\alpha h}{(\omega_0+\eta k_0^2)^2-\alpha^2k_0^2}, \tag{4.10}$$

$$\alpha = -\frac{1}{3}\frac{h(\omega_0+\eta k_0^2)+\alpha v^2k_0^2}{(\omega_0+\eta k_0^2)^2-\alpha^2k_0^2}. \tag{4.11}$$

If we set

$$h = xv^2k_0, \tag{4.12}$$

where $|x|\leqslant 1$, then the above equations become equivalent to

$$(\eta k_0^2+\alpha k_0)-\eta_0\,k_0^2 = \frac{v^2k_0^2}{3}\frac{1-x}{\omega_0+(\eta k_0^2+\alpha k_0)}, \tag{4.13}$$

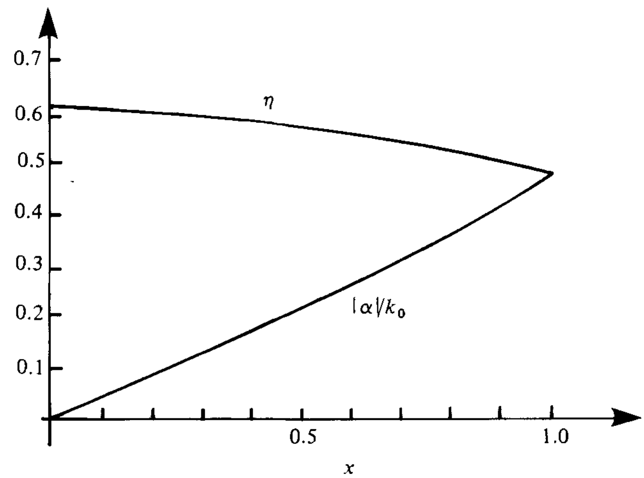

FIGURE 4. Variation of the self-consistent parameters η and α with helicity.

$$(\eta k_0^2 - \alpha k_0) - \eta_0 k_0^2 = \frac{v^2 k_0^2}{3} \frac{1+x}{\omega_0 + (\eta k_0^2 - \alpha k_0)}, \tag{4.14}$$

which may easily be solved for η and α.

When $x = 0$ and the helicity vanishes these equations reduce to $\alpha = 0$ and

$$\eta - \eta_0 = \frac{v^2}{3} \frac{1}{\omega_0 + \eta k_0^2}, \tag{4.15}$$

which is just the lowest-order version of the Phythian & Curtis equations for the effective diffusivity. When the helicity is maximal ($x = 1$) we find that

$$\alpha = -k_0(\eta - \eta_0), \tag{4.16}$$

$$\eta - \eta_0 = \frac{v^2}{3} \frac{1}{\omega_0 + \eta_0 k_0^2 + 2\eta k_0^2}. \tag{4.17}$$

Equation (4.16) corresponds to a result found by Kraichnan (1976*a*) in a different calculation.

In order to make a simple comparison we consider the case of high Péclet number and set $\eta_0 = 0$, $\omega_0 = k_0 = 1$, $v^2 = 3$. Then for $x = 0$ we find $\eta = 0.61$, $\alpha = 0$, while for $x = 1$ we have $\eta = 0.5$, $\alpha = -0.5$. In the magnetic case, then, the presence of helicity in the turbulence has its expected strong effect on the α-coefficient together with a small negative effect of the order of 20 % on the effective diffusivity. This is in contrast with the scalar-field case, where the effect of helicity is necessarily of higher order than the level of approximation considered here (Drummond 1982). A graph showing the continuous variation of η and α with x is given in figure 4. Clearly the behaviour is completely smooth. There is no sign of the dramatic reduction in the effective diffusivity encountered by Kraichnan (1976*b*) in his numerical simulation of magnetic diffusion. Kraichnan's results were particularly marked for frozen turbulence, which corresponds to $\omega_0 = 0$. Our results remain smooth in this limit. However, it may well be the case that a higher-order calculation will reveal new effects.

5. Conclusions

It is important to consider carefully the physical significance of the parameters we have calculated. Ideally we would like to have a complete parallel with the scalar-field case, where for large times the field on a sufficiently large spatial scale does obey an effective diffusion equation. The reason for this is that in the plane of the Laplace-transform variable the dominant singularity is a pole at $s = -\eta k^2$, where η is the effective diffusivity. The important region in transform space controlling the asymptotic behaviour is therefore $s \sim k^2 \sim 0$. This is just what happens in the magnetic case also when the helicity vanishes and $\alpha = 0$.

When helicity is present and $\alpha \neq 0$ the situation is changed. From (4.2)–(4.4) it is obvious that α and η are chosen so that

$$\overline{\mathscr{G}}_{ij}^{-1}(\boldsymbol{k}, s) = (s+\eta k^2)\,\delta_{ij} - \mathrm{i}\alpha\epsilon_{imj}\,k_m + O(sk, k^3, \ldots). \tag{5.1}$$

If we neglect the corrections then the dominant pole of $\overline{\mathscr{G}}_{ij}$ lies at

$$s - |\alpha|\,k - \eta k^2. \tag{5.2}$$

This should be most nearly correct for $k \ll k_0$, the scale of the turbulence. However, the behaviour at large times is dominated by the rightmost pole in the s-plane, which is located at

$$s = \frac{\alpha^2}{4\eta}, \tag{5.3}$$

when k has the value

$$k_c = \frac{|\alpha|}{2\eta}. \tag{5.4}$$

It follows that $\mathscr{G}_{ij}$ is dominated at large times by a term of the form

$$\mathscr{G}_{ij} \sim H_{ij}(\boldsymbol{x}-\boldsymbol{x}', t-t')\exp\{\alpha^2(t-t')/4\eta\}, \tag{5.5}$$

where H_{ij} is relatively slowly varying in time and has spatial oscillatory structure corresponding to wavenumber k_c.

However, this structure is only an acceptable consequence of the approximation if $k_c \ll k_0$. In fact this condition is violated for maximal helicity, when, according to our calculations, $|\alpha| = \eta k_0$ so that $k_c = \frac{1}{2}k_0$. It must be concluded therefore that the simple approximation to the propagator will only work uniformly in time and space either when $|\alpha|$ and therefore the helicity are well below their maximum values, or when the initial field distribution rigorously excludes large wavenumbers. It is true that in real turbulence there are mechanisms acting to suppress helicity (Moffatt 1979). Nevertheless the mathematical background is not as straightforward as the scalar-field case.

A further complication is associated with the fact that in calculating the small-k behaviour of the dominant s-plane pole it is not quite consistent to ignore the terms $O(sk)$ while keeping those $O(k^2)$. The magnitude of the correction can be calculated in our approximation, but even for maximal helicity does not produce a qualitative change in our conclusions.

There are two ways of viewing the correction. It can be regarded either as a further modification of the effective diffusivity or as necessitating an additional term in the effective partial differential equation for the magnetic field; thus

$$\frac{\partial}{\partial t}\boldsymbol{B} = \alpha\nabla \wedge \boldsymbol{B} + \xi\nabla \wedge \frac{\partial \boldsymbol{B}}{\partial t} - \eta\nabla \wedge (\nabla \wedge \boldsymbol{B}). \tag{5.6}$$

The difficulty with this equation is that it is rather singular because the operator

$$(\mathbb{1} - \xi \nabla \wedge) \tag{5.7}$$

has a zero eigenvalue for wavenumbers $k = \xi^{-1}$. If ξ were small this would be acceptable for small-wavenumber disturbances, but nevertheless it is difficult to use as part of a renormalized perturbation-theory scheme.

Although we rejected the detailed predictions of the model at large k, the suggestion that for large helicity the magnetic field develops a large-amplitude fine structure on the same scale as the turbulence may well have some truth in it. If so it suggests that there may be practical difficulties in making an accurate numerical simulation in this case. We note that maximal helicity was indeed the case studied by Kraichnan (1976*a*, *b*), so it would clearly be worthwhile attempting a simulation for a range of helicities from zero up to the maximum value. A possible explanation of the discrepancy between our results and his may then be revealed.

Finally we remark that, although we have used a rather artificial structure for the velocity correlation function in order to obtain simple analytic results, the theory could easily be applied to more-realistic choices. It remains to be shown, however, that the perturbative approach will work well when the k-dependence of the velocity spectrum has a power-law decrease rather than the sharp cutoff we have used here. In that case a different approach may be necessary (Moffatt 1981).

REFERENCES

DRUMMOND, I. T. 1982 Path integral methods for turbulent diffusion. *J. Fluid Mech.* **123**, 59.

KRAICHNAN, R. H. 1970 Diffusion by a random velocity field. *Phys. Fluids* **13**, 22.

KRAICHNAN, R. H. 1976*a* Diffusion of weak magnetic fields by isotropic turbulence. *J. Fluid Mech.* **75**, 657.

KRAICHNAN, R. H. 1976*b* Diffusion of passive-scalar and magnetic fields by helical turbulence. *J. Fluid Mech.* **77**, 753.

MOFFATT, H. K. 1979 *Magnetic Field Generation in Electrically Conducting Fluids*. Cambridge University Press.

MOFFATT, H. K. 1981 Some developments in the theory of turbulence. *J. Fluid Mech.* **106**, 2747.

ROBERTS, P. H. & STIX, M. 1971 *The Turbulent Dynamo: A Translation of a Series of papers by F. Krause, K.-H. Rädler and M. Steenbeck*. Technical Note IA-60 Boulder, Colorado.

PHYTHIAN, R. & CURTIS, W. D. 1978 The effective long-time diffusivity for a passive scalar field in a Gaussian model flow. *J. Fluid Mech.* **89**, 241.

STEENBECK, M., KRAUSE, F. & RÄDLER, K.-H. 1966 *Z. Naturforsch.* **21** a, 369–376. [English translation in Roberts & Stix 1971.]